$\mathrm{K}_1(\mathbb{S}_1)$ and the group of automorphisms of the algebra $\mathbb{S}_2$ of one-sided inverses of a polynomial algebra in two variables

V. V. Bavula

Abstract

Explicit generators are found for the group G_2 of automorphisms of the algebra $\mathbb{S}_2$ of one-sided inverses of a polynomial algebra in two variables over a field. Moreover, it is proved that

$$G_2 \simeq S_2 \ltimes \mathbb{T}^2 \ltimes \mathbb{Z} \ltimes ((K^* \ltimes E_\infty(\mathbb{S}_1)) \boxtimes_{\mathrm{GL}_\infty(K)} (K^* \ltimes E_\infty(\mathbb{S}_1)))$$

where S_2 is the symmetric group, $\mathbb{T}^2$ is the 2-dimensional algebraic torus, $E_\infty(\mathbb{S}_1)$ is the subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$ generated by the elementary matrices. In the proof, we use and prove several results on the index of operators, and the final argument in the proof is the fact that $\mathrm{K}_1(\mathbb{S}_1) \simeq K^*$ proved in the paper. The algebras $\mathbb{S}_1$ and $\mathbb{S}_2$ are noncommutative, non-Noetherian, and not domains. The group of units of the algebra $\mathbb{S}_2$ is found (it is huge).

Key Words: the group of automorphisms, the inner automorphisms, the Fredholm operators, the index of an operator, $\mathrm{K}_1(\mathbb{S}_1)$*, the semi-direct product of groups, the minimal primes.*

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1 Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers; K is a field and K^* is its group of units; $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\mathrm{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n and $\mathrm{Aut}_K(P_n)$ is its group of units (i.e. the group of all the invertible linear maps from P_n to P_n); the subalgebra $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ of $\mathrm{End}_K(P_n)$ is called the n'th *Weyl* algebra.

Definition, [4]. The *algebra* $\mathbb{S}_n$ *of one-sided inverses* of P_n is an algebra generated over a field K by $2n$ elements $x_1, \ldots, x_n, y_n, \ldots, y_n$ that satisfy the defining relations:

$$y_1x_1 = \cdots = y_nx_n = 1, \quad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \ \text{ for all } i \neq j,$$

where $[a, b] := ab - ba$, the commutator of elements a and b.

By the very definition, the algebra $\mathbb{S}_n$ is obtained from the polynomial algebra P_n by adding commuting, left (but not two-sided) inverses of its canonical generators. The algebra $\mathbb{S}_1$ is a well-known primitive algebra [8], p. 35, Example 2. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra* which is the $\mathbb{C}^*$-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y_1 = x_1^*$). The Toeplitz algebra is the universal $\mathbb{C}^*$-algebra generated by a proper isometry.

Example, [4]. Consider a vector space $V = \bigoplus_{i\in\mathbb{N}} Ke_i$ and two shift operators on V, $X : e_i \mapsto e_{i+1}$ and $Y : e_i \mapsto e_{i-1}$ for all $i \geq 0$ where $e_{-1} := 0$. The subalgebra of $\mathrm{End}_K(V)$ generated by the operators X and Y is isomorphic to the algebra $\mathbb{S}_1$ ($X \mapsto x$, $Y \mapsto y$). By taking the n'th tensor power $V^{\otimes n} = \bigoplus_{\alpha\in\mathbb{N}^n} Ke_\alpha$ of V we see that the algebra $\mathbb{S}_n \simeq \mathbb{S}_1^{\otimes n}$ is isomorphic to the subalgebra of $\mathrm{End}_K(V^{\otimes n})$ generated by the $2n$ shifts $X_1, Y_1, \ldots, X_n, Y_n$ that act in different directions.

Let $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ be the group of automorphisms of the algebra $\mathbb{S}_n$, and $\mathbb{S}_n^*$ be the group of units of the algebra $\mathbb{S}_n$.

Theorem 1.1 *1.* [5] $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.

2. [5] $G_1 = \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$.

where $S_n = \{s \in S_n \,|\, s(x_i) = x_{s(i)}, s(y_i) = y_{s(i)}\}$ is the symmetric group, $\mathbb{T}^n := \{t_\lambda \,|\, t_\lambda(x_i) = \lambda_i x_i, t_\lambda(y_i) = \lambda_i^{-1} y_i, \lambda = (\lambda_i) \in K^{*n}\}$ is the n-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{S}_n)$ is the group of inner automorphisms of the algebra $\mathbb{S}_n$ (which is huge), and $\mathrm{GL}_\infty(K)$ is the group of all the invertible infinite dimensional matrices of the type $1 + M_\infty(K)$ where the algebra (without 1) of infinite dimensional matrices $M_\infty(K) := \varinjlim M_d(K) = \bigcup_{d\geq 1} M_d(K)$ is the injective limit of matrix algebras. A semi-direct product $H_1 \ltimes H_2 \ltimes \cdots \ltimes H_m$ of several groups means that $H_1 \ltimes (H_2 \ltimes (\cdots \ltimes (H_{m-1} \ltimes H_m) \cdots))$.

The results of the papers [1, 4, 5, 6] and the present paper show that (when ignoring non-Noetherian property) the algebra $\mathbb{S}_n$ belongs to the family of algebras like the n'th Weyl algebra A_n, the polynomial algebra P_{2n} and the Jacobian algebra $\mathbb{A}_n$ (see [1, 6]). The structure of the group $G_1 = \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$ is another confirmation of 'similarity' of the algebras P_2, A_1, and $\mathbb{S}_1$. The groups of automorphisms of the polynomial algebra P_2 and the Weyl algebra A_1 (when $\mathrm{char}(K) = 0$) were found by Jung [10], Van der Kulk [11], and Dixmier [7] respectively. These two groups have almost identical structure, they are 'infinite GL-groups' in the sense that they are generated by the algebraic torus $\mathbb{T}^1$ and by the obvious automorphisms: $x \mapsto x + \lambda y^i$, $y \mapsto y$; $x \mapsto x$, $y \mapsto y + \lambda x^i$, where $i \in \mathbb{N}$ and $\lambda \in K$; which are sort of 'elementary infinite dimensional matrices' (i.e. 'infinite dimensional transvections'). The same picture as for the group G_1. In prime characteristic, the group of automorphism of the Weyl algebra A_1 was found by Makar-Limanov [9] (see also Bavula [3] for a different approach and for further developments).

Theorem 1.2 *1. $\mathbb{S}_n^* = K^* \times (1 + \mathfrak{a}_n)^*$ where the ideal $\mathfrak{a}_n$ of the algebra $\mathbb{S}_n$ is the sum of all the height 1 prime ideals of the algebra $\mathbb{S}_n$.*

2. The centre of the group $\mathbb{S}_n^*$ is K^*, and the centre of the group $(1 + \mathfrak{a}_n)^*$ is $\{1\}$.

3. The map $(1+\mathfrak{a}_n)^* \to \mathrm{Inn}(\mathbb{S}_n)$, $u \mapsto \omega_u$, is a group isomorphism ($\omega_u(a) := uau^{-1}$ for $a \in \mathbb{S}_n$).

The proof of this theorem is given at the end of Section 2 (another proof via the Jacobian algebras is given in [6]).

To save on notation, we identify the groups $(1 + \mathfrak{a}_n)^*$ and $\mathrm{Inn}(\mathbb{S}_n)$ via $u \mapsto \omega_u$. Clearly, $\mathbb{S}_2 = \mathbb{S}_1(1) \otimes \mathbb{S}_1(2)$ where $\mathbb{S}_1(i) := K\langle x_i, y_i\rangle \simeq \mathbb{S}_1$. The algebra $\mathbb{S}_2$ has only two height one prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Let $F_2 := \mathfrak{p}_1 \cap \mathfrak{p}_2$. The aim of the paper is to find generators for the group G_2 (see the end of the paper) and to prove the following theorem.

- (Theorem 2.13)
  1. $G_2 = S_2 \ltimes \mathbb{T}^2 \ltimes \Theta \ltimes ((U_1(K) \ltimes E_\infty(\mathbb{S}_1(2))) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(\mathbb{S}_1(1)))$.
  2. $G_2 \simeq S_2 \ltimes \mathbb{T}^2 \ltimes \mathbb{Z} \ltimes ((K^* \ltimes E_\infty(\mathbb{S}_1)) \boxtimes_{\mathrm{GL}_\infty(K)} (K^* \ltimes E_\infty(\mathbb{S}_1)))$.

For a ring R, let $E_\infty(R)$ be the subgroup of $\mathrm{GL}_\infty(R)$ generated by all the elementary matrices $1 + rE_{ij}$ (where $i, j \in \mathbb{N}$ with $i \neq j$, and $r \in R$), and let $U(R) := \{uE_{00} + 1 - E_{00} \,|\, u \in R^*\} \simeq R^*$ where R^* is the group of units of the ring R. The group $E_\infty(R)$ is a normal subgroup of $\mathrm{GL}_\infty(R)$. The group $\Theta \simeq \mathbb{Z}$ is generated by a single element θ (see Section 2).

If a group G is equal to the product $AB := \{ab \,|\, a \in A, b \in B\}$ of its *normal* subgroups A and B then we write $G = A \boxtimes B = A \boxtimes_{A\cap B} B$. So, each element $g \in G$ is a product ab for some elements $a \in A$ and $b \in B$; and $ab = a'b'$ (where $a' \in A$ and $b' \in B$) iff $a' = ac$ and $b' = c^{-1}b$ for some element $c \in A \cap B$. Clearly, $A \boxtimes B = B \boxtimes A$.

At the final stage of the proof of Theorem 2.13 we use the fact that

- (Theorem 2.11) $\mathrm{K}_1(\mathbb{S}_1) \simeq K^*$.

The group of units $\mathbb{S}_2^*$ of the algebra $\mathbb{S}_2$ is found.

- (Corollary 2.14) $\mathbb{S}_2^* = K^* \times \Theta \ltimes ((U_1(K) \ltimes E_\infty(\mathbb{S}_1(2)) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(\mathbb{S}_1(1)))$.

The structure of the proof of Theorem 2.13. The $\mathbb{S}_2$-module P_2 is faithful, and so $\mathbb{S}_2 \subseteq \mathrm{End}_K(P_2)$. By Theorem 1.1 and Theorem 1.2, the question of finding the group $G_2 = S_2 \ltimes \mathbb{T}^2 \ltimes (1+\mathfrak{a}_2)^*$ is equivalent to the question of finding the group $(1+\mathfrak{a}_2)^*$ or $\mathbb{S}_2^* = K^* \times (1+\mathfrak{a}_2)^*$. Difficulty in finding the group $\mathbb{S}_2^*$ stems from two facts: (i) $\mathbb{S}_2^* \subsetneqq \mathbb{S}_2 \cap \mathrm{Aut}_K(P_2)$, i.e. there are non-units of the algebra $\mathbb{S}_2$ that are invertible linear maps in P_2; and (ii) some units of the algebra $\mathbb{S}_2$ are product of *non-units* with *non-zero* indices (each unit has zero index). To eliminate (ii) the group Θ is introduced, and it is proved that $(1+\mathfrak{a}_2)^* = \Theta \ltimes \mathcal{K}$ and for the normal subgroup $\mathcal{K}$ of $(1+\mathfrak{a}_2)^*$ the situation (ii) does not occur. The group $\mathcal{K}$ is the common kernel of group epimorphisms $\mathrm{ind}_i : (1+\mathfrak{a}_2)^* \to \mathbb{Z}$ (see (9)) where $i = 1, 2$. In order to construct the maps ind_i and to prove that they are well defined group homomorphisms we need several results on the index of operators which are collected at the beginning of Section 2. Some of these are new (Theorem 2.3 and Corollary 2.5). Briefly, using indices of operators is the main tool in finding the group $\mathbb{S}_2^*$ and to prove that $\mathrm{K}_1(\mathbb{S}_1) \simeq K^*$. Using indices and the fact that $(1+F_2)^* = (1+F_2) \cap \mathrm{Aut}_K(P_2)$ we show that $\mathcal{K} = (1+\mathfrak{p}_1)^* \boxtimes_{(1+F_2)^*} (1+\mathfrak{p}_2)^*$ (Proposition 2.9). Then using indices, we prove that $(1+\mathfrak{p}_i)^* = U_i(K) \ltimes E_\infty(\mathbb{S}_1(i+1))$ (Proposition 2.10). This fact is equivalent to the fact that $\mathrm{K}_1(\mathbb{S}_1) \simeq K^*$ (Theorem 2.11).

2 The groups G_2 and $\mathrm{K}_1(\mathbb{S}_1)$

In this section, the groups G_2, $\mathbb{S}_2^*$, and and $K_1(\mathbb{S}_1)$ are found (Theorems 2.13 and 2.11, Corollary 2.14). The proofs are constructive.

We mentioned already in the Introduction that the key idea in finding the group G_2 is to use indices of operators. That is why we start this section with collecting known results on indices and prove new ones. These results are used in all the proofs that follow.

The index ind **of linear maps and its properties**. Let $\mathcal{C}$ be the family of all K-linear maps with finite dimensional kernel and cokernel, i.e. $\mathcal{C}$ is the family of **Fredholm** linear maps/operators. For vector spaces V and U, let $\mathcal{C}(V,U)$ be the set of all the linear maps from V to U with finite dimensional kernel and cokernel. So, $\mathcal{C} = \bigcup_{V,U} \mathcal{C}(V,U)$ is the disjoint union.

Definition. For a linear map $\varphi \in \mathcal{C}$, the integer $\mathrm{ind}(\varphi) := \dim \ker(\varphi) - \dim \mathrm{coker}(\varphi)$ is called the **index** of the map φ.

For vector spaces V and U, let $\mathcal{C}(V,U)_i := \{\varphi \in \mathcal{C}(V,U) \mid \mathrm{ind}(\varphi) = i\}$. Then $\mathcal{C}(V,U) = \bigcup_{i\in\mathbb{Z}} \mathcal{C}(V,U)_i$ is the disjoint union, and the family $\mathcal{C}$ is the disjoint union $\bigcup_{i\in\mathbb{Z}} \mathcal{C}_i$ where $\mathcal{C}_i := \{\varphi \in \mathcal{C} \mid \mathrm{ind}(\varphi) = i\}$. When $V = U$, we write $\mathcal{C}(V) := \mathcal{C}(V,V)$ and $\mathcal{C}(V)_i := \mathcal{C}(V,V)_i$.

Example. Note that $\mathbb{S}_1 \subset \mathrm{End}_K(P_1)$ ($x * x^i = x^{i+1}$, $y * x^{i+1} = x^i$, $i \in \mathbb{N}$, and $y * 1 = 0$). The map x^i acting on the polynomial algebra P_1 is an injection with $P_1 = (\bigoplus_{j=0}^{i-1} Kx^j) \bigoplus \mathrm{im}(x^i)$; and the map y^i acting on P_1 is a surjection with $\ker(y^i) = \bigoplus_{j=0}^{i-1} Kx^j$, and so

$$\mathrm{ind}(x^i) = -i \text{ and } \mathrm{ind}(y^i) = i, \quad i \geq 1. \tag{1}$$

Lemma 2.1 shows that $\mathcal{C}$ is a multiplicative semigroup with zero element (if the composition of two elements of C is undefined we set their product to be zero).

Lemma 2.1 *Let $\psi : M \to N$ and $\varphi : N \to L$ be K-linear maps. If two of the following three maps: ψ, φ, and $\varphi\psi$, belong to the set $\mathcal{C}$ then so does the third; and in this case,*

$$\mathrm{ind}(\varphi\psi) = \mathrm{ind}(\varphi) + \mathrm{ind}(\psi).$$

By Lemma 2.1, $\mathcal{C}(N,L)_i\mathcal{C}(M,N)_j \subseteq \mathcal{C}(M,L)_{i+j}$ for all $i, j \in \mathbb{Z}$.

Lemma 2.2 *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of K-linear maps with exact rows. Suppose that $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{C}$. Then

$$\mathrm{ind}(\varphi_2) = \mathrm{ind}(\varphi_1) + \mathrm{ind}(\varphi_3).$$

Let V and U be vector spaces. Define $\mathcal{I}(V,U) := \{\varphi \in \mathrm{Hom}_K(V,U) \,|\, \dim \mathrm{im}(\varphi) < \infty\}$, and when $V = U$ we write $\mathcal{I}(V) := \mathcal{I}(V,V)$.

Theorem 2.3 *Let V and U be vector spaces. Then $\mathcal{C}(V,U)_i + \mathcal{I}(V,U) = \mathcal{C}(V,U)_i$ for all $i \in \mathbb{Z}$.*

Proof. The theorem is obvious if the vector spaces V and U are finite dimensional since in this case the index of each linear map from V to U is equal to $\dim(V) - \dim(U)$. We deduce the general case from this one. Let $u \in \mathcal{C}(V,U)_i$ and $f \in \mathcal{I}(V,U)$. Using the fact that the kernel $\ker(f)$ has finite codimension in V, i.e. $\dim(V/\ker(f)) < \infty$ (since $V/\ker(f) \simeq \mathrm{im}(f)$), we can easily find subspaces $V_1, V_2 \subseteq V$ and $W, U_1, U_2 \subseteq U$ such that $\dim(V_1) < \infty$, $\dim(U_1) < \infty$, $\dim(W) < \infty$,

$$V = \ker(u) \bigoplus V_1 \bigoplus V_2, \; U = W \bigoplus U_1 \bigoplus U_2, \; u|_{V_1} : V_1 \simeq U_1, \; u|_{V_2} : V_2 \simeq U_2, \; f(V_2) = 0$$

and $f(\ker(u) \bigoplus V_1) \subseteq W \bigoplus U_1$. Note that $\mathrm{im}(u) = U_1 \bigoplus U_2$ and $U/\mathrm{im}(u) \simeq W$. Consider the restrictions, say u' and f', of the maps u and f to the finite dimensional subspace $\ker(u) \bigoplus V_1$ of V, i.e.

$$u', f' : \ker(u) \bigoplus V_1 \to W \bigoplus U_1. \tag{2}$$

Then it is obvious that $\mathrm{ind}(u') = \mathrm{ind}(u)$ and $\mathrm{ind}(u'+f') = \mathrm{ind}(u+f)$ (since $u+f|_{V_2} = u|_{V_2} : V_2 \simeq U_2$). On the other hand, $\mathrm{ind}(u'+f') = \mathrm{ind}(u')$ since the vector spaces in (2) are finite dimensional. Therefore, $\mathrm{ind}(u+f) = \mathrm{ind}(u)$. $\square$

Lemma 2.4 *Let V and V' be vector spaces, and $\varphi : V \to V'$ be a linear map such that the vector spaces $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ are isomorphic. Fix subspaces $U \subseteq V$ and $W \subseteq V'$ such that $V = \ker(\varphi) \bigoplus U$ and $V' = W \bigoplus \mathrm{im}(\varphi)$ and fix an isomorphism $f : \ker(\varphi) \to W$ (this is possible since $\ker(\varphi) \simeq \mathrm{coker}(\varphi) \simeq W$) and extend it to a linear map $f : V \to V'$ by setting $f(U) = 0$. Then the map $\varphi + f : V \to V'$ is an isomorphism.*

Proof. The map $\varphi + f$ is a surjection since $(\varphi+f)(V) = (\varphi+f)(\ker(\varphi)+U) = W + \mathrm{im}(\varphi) = V'$. The map $\varphi + f$ is an injection: if $v \in \ker(\varphi+f)$ then $\varphi(v) = f(-v) \in W \cap \mathrm{im}(\varphi) = 0$, and so $v \in \ker(\varphi) \cap \ker(f) = \ker(\varphi) \cap U = 0$. Therefore, the map $\varphi + f$ is an isomorphism. $\square$

Lemma 2.5 *Let V and V' be vector spaces, $\varphi \in \mathcal{C}(V,V')_i$ for some $i \in \mathbb{Z}$, $V = \ker(\varphi) \bigoplus U$ and $V' = W \bigoplus \mathrm{im}(\varphi)$ for some subspaces $U \subseteq V$ and $W \subseteq V'$.*

1. *If $\dim \ker(\varphi) \leq \dim \mathrm{coker}(\varphi)$ then fix an injective linear map $f : \ker(\varphi) \to W$ and extend it to a linear map $f : V \to V'$ by setting $f(U) = 0$. Then the map $\varphi + f$ is an injection that belongs to $\mathcal{C}(V,V')_i$.*

2. *If $\dim \ker(\varphi) \geq \dim \mathrm{coker}(\varphi)$ then fix a surjective linear map $f : \ker(\varphi) \to W$ and extend it to a linear map $f : V \to V'$ by setting $f(U) = 0$. Then the map $\varphi + f$ is a surjection that belongs to $\mathcal{C}(V,V')_i$.*

Proof. 1. An arbitrary element $v \in V = \ker(\varphi) \bigoplus U$ is a unique sum $k + u$ where $k \in \ker(\varphi)$ and $u \in U$. If $v \in \ker(\varphi+f)$ then $0 = (\varphi+f)(k+u) = f(k) + \varphi(u)$ and so $f(k) = 0$ and $\varphi(u) = 0$ (since $V' = W \bigoplus \mathrm{im}(\varphi)$, $f(k) \in W$ and $\varphi(u) \in \mathrm{im}(\varphi)$) hence $k = 0$ (f is an injection) and $u = 0$

$(\varphi|_U : U \simeq \mathrm{im}(\varphi))$. Therefore, $v = 0$. This means that the map $\varphi + f$ is an injection. Now, $\mathrm{im}(\varphi + f) = f(\ker(\varphi)) \bigoplus \mathrm{im}(\varphi)$, and so

$$\mathrm{ind}(\varphi + f) = -\dim \mathrm{coker}(\varphi + f) = -\dim(W/\ker(\varphi)) = \dim \ker(\varphi) - \dim(W) = \mathrm{ind}(\varphi).$$

2. The map $\varphi+f$ is a surjection since $(\varphi+f)(V) = (\varphi+f)(\ker(\varphi)+U) = f(\ker(\varphi))+\mathrm{im}(\varphi) = W + \mathrm{im}(\varphi) = V'$. Suppose that an element $v = k + u \in V$ (where $k \in \ker(\varphi)$ and $u \in U$) belongs to the kernel of the map $\varphi + f$. Then $0 = (\varphi + f)(v) = f(k) + \varphi(u)$ and so $f(k) = 0$ and $\varphi(u) = 0$ (since $V' = W \bigoplus \mathrm{im}(\varphi)$, $f(k) \in W$ and $\varphi(u) \in \mathrm{im}(U)$), hence $k \in \ker(f)$ and $u = 0$ since $(\varphi|_U : U \simeq \mathrm{im}(\varphi))$. Now,

$$\mathrm{ind}(\varphi + f) = \dim \ker(f) = \dim \ker(\varphi) - \dim(W) = \mathrm{ind}(\varphi). \quad \Box$$

The algebras $\mathbb{S}_1$ and $\mathbb{S}_2$. We collect some results without proofs on the algebras $\mathbb{S}_1$ and $\mathbb{S}_2$ from [4] that will be used in this paper, their proofs can be found in [4]. Clearly, $\mathbb{S}_2 = \mathbb{S}_1(1) \otimes \mathbb{S}_1(2) \simeq \mathbb{S}_1^{\otimes 2}$ where $\mathbb{S}_1(i) := K\langle x_i, y_i \,|\, y_i x_i = 1\rangle \simeq \mathbb{S}_1$ and $\mathbb{S}_2 = \bigoplus_{\alpha,\beta\in\mathbb{N}^2} Kx^\alpha y^\beta$ where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2)$, $y^\beta := y_1^{\beta_1} y_2^{\beta_2}$, $\beta = (\beta_1, \beta_2)$. In particular, the algebra $\mathbb{S}_2$ contains two polynomial subalgebras P_2 and $Q_2 := K[y_1, y_2]$ and is equal, as a vector space, to their tensor product $P_2 \otimes Q_2$.

When $n = 1$, we usually drop the subscript '1' if this does not lead to confusion. So, $\mathbb{S}_1 = K\langle x, y \,|\, yx = 1\rangle = \bigoplus_{i,j\geq 0} Kx^i y^j$. For each natural number $d \geq 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} KE_{ij}$ be the algebra of d-dimensional matrices where $\{E_{ij}\}$ are the matrix units, $M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$ be the algebra (without 1) of infinite dimensional matrices, and $\mathrm{GL}_\infty(K)$ be the group of units of the monoid $1 + M_\infty(K)$. The algebra $\mathbb{S}_1$ contains the ideal $F := \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$, where

$$E_{ij} := x^i y^j - x^{i+1} y^{j+1}, \;\; i, j \geq 0. \tag{3}$$

For all natural numbers i, j, k, and l, $E_{ij} E_{kl} = \delta_{jk} E_{il}$ where δ_{jk} is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$.

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F, \tag{4}$$

the direct sum of vector spaces. Then

$$\mathbb{S}_1/F \simeq K[x, x^{-1}] =: L_1, \;\; x \mapsto x, \;\; y \mapsto x^{-1}, \tag{5}$$

since $yx = 1$, $xy = 1 - E_{00}$ and $E_{00} \in F$. The algebra $\mathbb{S}_2 = \mathbb{S}_1(1) \otimes \mathbb{S}_1(2)$ contains the ideal

$$F_2 := F(1) \otimes F(2) = \bigoplus_{\alpha,\beta\in\mathbb{N}^2} KE_{\alpha\beta}, \;\; \text{where} \;\; E_{\alpha\beta} := \prod_{i=1}^{2} E_{\alpha_i\beta_i}(i),$$

where $F(i)$ is the ideal F of the algebra $\mathbb{S}_1(i)$ and $E_{\alpha_i\beta_i}(i)$ are its matrix units as defined in (3). Note that $E_{\alpha\beta} E_{\gamma\rho} = \delta_{\beta\gamma} E_{\alpha\rho}$ for all elements $\alpha, \beta, \gamma, \rho \in \mathbb{N}^2$ where $\delta_{\beta\gamma}$ is the Kronecker delta function, and so $(1 + F_2)^* \simeq \mathrm{GL}_\infty(K)$.

The algebra $\mathbb{S}_2$ contains only two height one prime ideals $\mathfrak{p}_1 := F(1) \otimes \mathbb{S}_1(2)$ and $\mathfrak{p}_2 := \mathbb{S}_1(1) \otimes F(2)$. Clearly, $F_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1 \mathfrak{p}_2$, $\mathbb{S}_2/\mathfrak{p}_1 \simeq K[x_1, x_1^{-1}] \otimes \mathbb{S}_1(2)$ and $\mathbb{S}_2/\mathfrak{p}_2 \simeq \mathbb{S}_1(1) \otimes K[x_2, x_2^{-1}]$. The ideal $\mathfrak{a}_2 := \mathfrak{p}_1 + \mathfrak{p}_2$ plays an important role in this paper, $\mathbb{S}_2/\mathfrak{a}_2 \simeq K[x_1, x_1^{-1}] \otimes K[x_2, x_2^{-1}]$.

Proposition 2.6 [4] *The polynomial algebra P_n is the only (up to isomorphism) faithful, simple $\mathbb{S}_n$-module.*

In more detail, ${}_{\mathbb{S}_n}P_n \simeq \mathbb{S}_n/(\sum_{i=0}^n \mathbb{S}_n y_i) = \bigoplus_{\alpha\in\mathbb{N}^n} Kx^\alpha \overline{1}$, $\overline{1} := 1 + \sum_{i=1}^n \mathbb{S}_n y_i$; and the action of the canonical generators of the algebra $\mathbb{S}_n$ on the polynomial algebra P_n is given by the rule:

$$x_i * x^\alpha = x^{\alpha+e_i}, \;\; y_i * x^\alpha = \begin{cases} x^{\alpha-e_i} & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0, \end{cases} \;\; \text{and} \;\; E_{\beta\gamma} * x^\alpha = \delta_{\gamma\alpha} x^\beta,$$

where the set of elements $e_1 := (1, 0, \dots, 0), \dots, e_n := (0, \dots, 0, 1)$ is the canonical basis for the free $\mathbb{Z}$-module $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. We identify the algebra $\mathbb{S}_n$ with its image in the algebra $\mathrm{End}_K(P_n)$ of all the K-linear maps from the vector space P_n to itself, i.e. $\mathbb{S}_n \subset \mathrm{End}_K(P_n)$.

Corollary 2.7 *1.* $1 + F_2 \subseteq \mathcal{C}(P_2)_0$.

2. $\mathbb{S}_2^* + F_2 \subseteq \mathcal{C}(P_2)_0$.

Proof. Both statements follows from Theorem 2.3: $\mathbb{S}_2^* \in \mathcal{C}(P_2)_0$ and $F_2 \in \mathcal{I}(P_2)$, but we give short independent proofs (that do not use Theorem 2.3).

1. Since $1 + F_2 \simeq 1 + M_\infty(K)$, statement 1 is obvious.

2. Let $u \in \mathbb{S}_2^*$ and $f \in F_2$. Then $u^{-1}f \in F_2$. By statement 1, the element $1 + u^{-1}f \in \mathcal{C}(P_2)_0$. Since $u \in \mathcal{C}(P_2)_0$, we have $u + f = u(1 + u^{-1}f) \in \mathcal{C}(P_2)_0$, by Lemma 2.1. $\Box$

The subgroup Θ of $(1 + \mathfrak{a}_2)^*$. The element $\theta := (1 + (y_1 - 1)E_{00}(2))(1 + E_{00}(1)(x_2 - 1)) \in (1 + \mathfrak{a}_2)^*$ is a unit and

$$\theta^{-1} = (1 + E_{00}(1)(y_2 - 1))(1 + (x_1 - 1)E_{00}(2)) \in (1 + \mathfrak{a}_2)^*. \tag{6}$$

This is obvious since

$$\theta * x^\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 > 0, \alpha_2 > 0, \\ x_1^{\alpha_1 - 1} & \text{if } \alpha_1 > 0, \alpha_2 = 0, \\ x_2^{\alpha_2 + 1} & \text{if } \alpha_1 = 0, \alpha_2 \geq 0, \end{cases} \quad \text{and} \quad \theta^{-1} * x^\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 > 0, \alpha_2 > 0, \\ x_1^{\alpha_1 + 1} & \text{if } \alpha_1 \geq 0, \alpha_2 = 0, \\ x_2^{\alpha_2 - 1} & \text{if } \alpha_1 = 0, \alpha_2 > 0. \end{cases}$$

Let Θ be the subgroup of $(1 + \mathfrak{a}_2)^*$ generated by the element θ. Then $\Theta \simeq \mathbb{Z}$ since $\theta^i * 1 = x_2^i$ for all $i \geq 1$. It follows from

$$(1 + (y_1 - 1)E_{00}(2)) * x^\alpha = \begin{cases} x^\alpha & \text{if } \alpha_2 > 0, \\ x_1^{\alpha_1 - 1} & \text{if } \alpha_1 > 0, \alpha_2 = 0, \\ 0 & \text{if } \alpha_1 = 0, \alpha_2 = 0, \end{cases}$$

that the map $1 + (y_1 - 1)E_{00}(2) \in \mathrm{End}_K(P_2)$ is a surjection with kernel equal to K, and so

$$\mathrm{ind}(1 + (y_1 - 1)E_{00}(2)) = 1. \tag{7}$$

Similarly, it follows from

$$(1 + E_{00}(1)(x_2 - 1)) * x^\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 > 0, \\ x_2^{\alpha_2 + 1} & \text{if } \alpha_1 = 0, \end{cases}$$

that the map $1 + E_{00}(1)(x_2 - 1) \in \mathrm{End}_K(P_2)$ is an injection with $P_2 = K \bigoplus \mathrm{im}(1 + E_{00}(1)(x_2 - 1))$, and so

$$\mathrm{ind}(1 + E_{00}(1)(x_2 - 1)) = -1. \tag{8}$$

We see that the unit θ of the algebra $\mathbb{S}_2$ is the product of two *non-units* having nonzero indices the sum of which is equal to zero since $\mathrm{ind}(\theta) = 0$. Lemma 2.8 shows that this is a general phenomenon, and so the group $(1 + \mathfrak{a}_2)^*$ is a sophisticated group in the sense that in construction of units non-units are involved.

Lemma 2.8 *Let* $u = 1 + a_1 + a_2 \in (1 + \mathfrak{a}_2)^*$ *where* $a_i \in \mathfrak{p}_i$. *Then*

1. $1 + a_1, 1 + a_2 \in \mathcal{C}(P_2)$ *and* $\mathrm{ind}(1 + a_1) + \mathrm{ind}(1 + a_2) = 0$.

2. If $u = 1 + a_1' + a_2'$ *where* $a_i' \in \mathfrak{p}_i$ *then* $\mathrm{ind}(1 + a_1) = \mathrm{ind}(1 + a_1')$ *and* $\mathrm{ind}(1 + a_2) = \mathrm{ind}(1 + a_2')$.

Proof. 1. Since $a_i \in \mathfrak{p}_i$, we have $a_1a_2, a_2a_1 \in F_2$. By Corollary 2.7.(2), $u + a_1a_2, u + a_2a_1 \in \mathcal{C}(P_2)_0$. Then, it follows from the equalities $u+a_1a_2 = (1+a_1)(1+a_2)$ and $u+a_2a_1 = (1+a_2)(1+a_1)$, that

$$\begin{aligned} \mathrm{im}(1+a_1) &\supseteq \mathrm{im}(u+a_1a_2), \quad \ker(1+a_1) \subseteq \ker(u+a_2a_1), \\ \mathrm{im}(1+a_2) &\supseteq \mathrm{im}(u+a_2a_1), \quad \ker(1+a_2) \subseteq \ker(u+a_1a_2). \end{aligned}$$

This means that $1+a_1, 1+a_2 \in \mathcal{C}(P_2)$. By Corollary 2.7.(2) and Lemma 2.1,

$$0 = \mathrm{ind}(u+a_1a_2) = \mathrm{ind}(1+a_1)(1+a_2) = \mathrm{ind}(1+a_1) + \mathrm{ind}(1+a_2).$$

2. It is obvious that $a' = a_1 + f$ and $a_2' = a_2 - f$ for an element $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 = F_2$. Since $F_2 \subseteq \mathcal{I}(P_2)$, we see that $\mathrm{ind}(1+a_1') = \mathrm{ind}(1+a_1+f) = \mathrm{ind}(1+a_1)$ and $\mathrm{ind}(1+a_2') = \mathrm{ind}(1+a_2-f) = \mathrm{ind}(1+a_2)$, by Theorem 2.3. $\Box$

By Lemma 2.8, for each number $i = 1, 2$, there is a well-defined map,

$$\mathrm{ind}_i : (1+\mathfrak{a}_2)^* \to \mathbb{Z}, \quad u = 1 + a_1 + a_2 \mapsto \mathrm{ind}(1+a_i), \tag{9}$$

which is a group homomorphism:

$$\begin{aligned} \mathrm{ind}_i(uu') &= \mathrm{ind}_i((1+a_1+a_2)(1+a_1'+a_2')) = \mathrm{ind}(1+a_i+a_i'+a_ia_i') \\ &= \mathrm{ind}((1+a_i)(1+a_i')) = \mathrm{ind}(1+a_i) + \mathrm{ind}(1+a_i') \\ &= \mathrm{ind}_i(u) + \mathrm{ind}_i(u'). \end{aligned}$$

The restriction of the homomorphism ind_i to the subgroup Θ is an isomorphism: $\mathrm{ind}_1 : \Theta \to \mathbb{Z}$, $\theta \mapsto -1$; $\mathrm{ind}_2 : \Theta \to \mathbb{Z}$, $\theta \mapsto 1$. Therefore, the homomorphisms ind_i are epimorphisms which have the *same* kernel (Lemma 2.8.(1)) which we denote by $\mathcal{K}$. Then,

$$(1+\mathfrak{a}_2)^* = \Theta \ltimes \mathcal{K}. \tag{10}$$

It is obvious that $(1+\mathfrak{p}_1)^* \boxtimes_{(1+F_2)^*} (1+\mathfrak{p}_2)^* \subseteq \mathcal{K}$ since $(1+\mathfrak{p}_1)^* \cap (1+\mathfrak{p}_2)^* = (1+F_2)^*$.

Proposition 2.9 $\mathcal{K} = (1+\mathfrak{p}_1)^* \boxtimes_{(1+F_2)^*} (1+\mathfrak{p}_2)^*$.

Proof. It suffices to show that each element $u = 1 + a_1 + a_2$ of the group $\mathcal{K}$ is a product u_1u_2 for some elements $u_i \in (1+\mathfrak{p}_i)^*$. Note that $1 + a_1 \in \mathcal{C}(P_2)_0$. Fix a subspace, say W, of P_2 such that $P_2 = \ker(1+a_1) \bigoplus W$ and $W = \bigoplus_{\alpha \in I} Kx^\alpha$ where I is a subset of $\mathbb{N}^2$. By Lemma 2.4, we can find an element $f_1 \in F_2$ (since $\dim \ker(1+a_1) < \infty$, W has a monomial basis, and $f_1(W) = 0$) such that $u_1 := 1 + a_1 + f_1 \in \mathrm{Aut}_K(P_2)$. We claim that $u_1 \in (1+\mathfrak{p}_1)^*$. It is a subtle point since not all elements of the algebra $\mathbb{S}_2$ that are invertible linear maps in P_2 are invertible in $\mathbb{S}_2$, i.e. $\mathbb{S}_2^* \subsetneqq \mathbb{S}_2 \cap \mathrm{Aut}_K(P_2)$ but $(1+F_2)^* = (1+F_2) \cap \mathrm{Aut}_K(P_2)$, [5]. The main idea in the proof of the claim is to use this equality. Similarly, we can find an element $f_2 \in F_2$ such that $v := 1 + a_2 + f_2 \in \mathrm{Aut}_K(P_2)$. Then $u = u_1v + g_1$ and $u = vu_1 + g_2$ for some elements $g_i \in F_2$. Hence,

$$u_1vu^{-1} = 1 - g_1u^{-1} \quad \text{and} \quad u^{-1}vu_1 = 1 - u^{-1}g_2,$$

and so $1 - g_1u^{-1}, 1 - u^{-1}g_2 \in (1+F_2) \cap \mathrm{Aut}_K(P_2) = (1+F_2)^*$. It follows that $u_1^{-1} = vu^{-1}(1 - g_1u^{-1})^{-1} \in (1+\mathfrak{p}_1)^*$ since

$$1 \equiv 1 - g_1u^{-1} \equiv u_1vu^{-1} \equiv vu^{-1} \mod \mathfrak{p}_1.$$

This proves the claim. Clearly, $u_2 := v + u_1^{-1}g_1 \in 1 + \mathfrak{p}_2$. Then, it follows from the equality $u = u_1v + g_1 = u_1(v + u_1^{-1}g_1) = u_1u_2$ that $u_2 = u_1^{-1}u \in (1+\mathfrak{p}_2)^*$. This finishes the proof of the proposition. $\Box$

In order to save on notation, it is convenience to treat the set of indices $\{1,2\}$ as the group $\mathbb{Z}/2\mathbb{Z}=\{1,2\}$ where $1+1=2$ and $1+2=1$. For each number $i=1,2$, the group of units of the monoid $1+\mathfrak{p}_i=1+F(i)\otimes\mathbb{S}_1(i+1)=1+M_\infty(\mathbb{S}_1(i+1))$ is equal to $(1+\mathfrak{p}_i)^*=\mathrm{GL}_\infty(\mathbb{S}_1(i+1))$. It contains the semi-direct product $U_i(K)\ltimes E_\infty(\mathbb{S}_1(i+1))$ of its two subgroups, where

$$U_i(K):=\{\lambda E_{00}(i)+1-E_{00}(i)\,|\,\lambda\in K^*\}\simeq K^*$$

and the group $E_\infty(\mathbb{S}_1(i+1))$ is generated by all the elementary matrices $1+aE_{kl}(i)$ where $k\neq l$ and $a\in\mathbb{S}_1(i+1)$. Note that the group $E_\infty(\mathbb{S}_1(i+1))$ is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1(i+1))$.

The set F_2 is an ideal of the algebra $K+\mathfrak{p}_i=K(1+\mathfrak{p}_i)$ which is a subalgebra of the algebra $\mathbb{S}_2$, and $(K+\mathfrak{p}_i)/F_2=K(1+\mathfrak{p}_i/F_2)\simeq K(1+M_\infty(L_{i+1}))$ where $L_{i+1}:=K[x_{i+1},x_{i+1}^{-1}]\simeq\mathbb{S}_1(i+1)/F(i+1)$ is the Laurent polynomial algebra. The algebra L_{i+1} is a Euclidian domain, hence $\mathrm{GL}_\infty(L_{i+1})=U(L_{i+1})\ltimes E_\infty(L_{i+1})$ where

$$U(L_{i+1}):=\{aE_{00}(i)+1-E_{00}(i)\,|\,a\in L_{i+1}^*\}\simeq L_{i+1}^*=K^*\times\{x_{i+1}^m\,|\,m\in\mathbb{Z}\}$$

and $E_\infty(L_{i+1})$ is the subgroup of $\mathrm{GL}_\infty(L_{i+1})$ generated by all the elementary matrices.

The group of units of the algebra $(K+\mathfrak{p}_i)/F_2$ is equal to $K^*\times\mathrm{GL}_\infty(L_{i+1})=K^*\times(U(L_{i+1})\ltimes E_\infty(L_{i+1}))$. The algebra epimorphism $\psi_i:K+\mathfrak{p}_i\to(K+\mathfrak{p}_i)/F_2$, $a\mapsto a+F_2$, induces the exact sequence of groups,

$$1\to(1+F_2)^*\to(1+\mathfrak{p}_i)^*\stackrel{\psi_i}{\to}\mathrm{GL}_\infty(L_{i+1})=U(L_{i+1})\ltimes E_\infty(L_{i+1}),\tag{11}$$

which yields the short exact sequence of groups,

$$1\to(1+F_2)^*\to U_i(K)\ltimes E_\infty(\mathbb{S}_1(i+1))\to U(K)\ltimes E_\infty(L_{i+1})\to 1\tag{12}$$

since $(1+F_2)^*\subseteq E_\infty(\mathbb{S}_1(i+1))$, by Proposition 2.15. Note that $U_i(K)\ltimes E_\infty(\mathbb{S}_1(i+1))\subseteq(1+\mathfrak{p}_i)^*$. In fact, the equality holds.

Proposition 2.10 $(1+\mathfrak{p}_i)^*=U_i(K)\ltimes E_\infty(\mathbb{S}_1(i+1))$.

Proof. In view of the exact sequences (11) and (12), it suffices to show that the image of the map ψ_i in (11) is equal to $U(K)\ltimes E_\infty(L_{i+1}))$. Since

$$U(L_{i+1})=U(K)\times\{E_{00}(i)x_{i+1}^m+1-E_{00}(i)\,|\,m\in\mathbb{Z}\},$$

this is equivalent to show that that if $\psi_i(u)=E_{00}(i)x_{i+1}^m+1-E_{00}(i)$ for some element $u\in(1+\mathfrak{p}_i)^*$ and an integer $m\in\mathbb{Z}$ then $m=0$. Let $u(m):=E_{00}(i)v_{i+1}(m)+1-E_{00}(i)$ where $v_{i+1}(m):=\begin{cases}x_{i+1}^m & \text{if } m\geq 0,\\ y_{i+1}^{|m|} & \text{if } m<0.\end{cases}$ Then $u(m)\in 1+\mathfrak{p}_i$ and $\psi_i(u(m))=\psi_i(u)$. Hence, $u(m)=u+f_m$ for some element $f_m\in F_2$. Note that

$$u(m)=\begin{cases}u(1)^m & \text{if } m\geq 0,\\ u(-1)^{|m|} & \text{if } m<0,\end{cases}$$

and, by (7) and (8), $\mathrm{ind}(u(m))=-m$. By Corollary 2.7.(2).

$$0=\mathrm{ind}(u)=\mathrm{ind}(u+f_m)=\mathrm{ind}(u(m))=-m,$$

and so $m=0$, as required. $\Box$

Proposition 2.10 is equivalent to the next theorem.

Theorem 2.11 $\mathrm{K}_1(\mathbb{S}_1)\simeq U(K)\simeq K^*$ *and* $\mathrm{GL}_\infty(\mathbb{S}_1)=U(K)\ltimes E_\infty(\mathbb{S}_1)$.

Proof. Recall that $\mathrm{K}_1(\mathbb{S}_1) := \mathrm{GL}_\infty(\mathbb{S}_1)/E_\infty(\mathbb{S}_1)$ where $E_\infty(\mathbb{S}_1)$ is the subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$ generated by the elementary matrices. The group $E_\infty(\mathbb{S}_1)$ is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$. It follows from $(1+\mathfrak{p}_i)^* \simeq \mathrm{GL}_\infty(\mathbb{S}_1)$ and Proposition 2.10 that

$$\mathrm{K}_1(\mathbb{S}_1) \simeq U(K) \ltimes E_\infty(\mathbb{S}_1)/E_\infty(\mathbb{S}_1) \simeq U(K) \simeq K^*. \quad \square$$

The determinant $\overline{\det}$. The algebra epimorphism $\mathbb{S}_1 \to \mathbb{S}_1/F \simeq L_1$, $a \mapsto a+F$, yields the group homomorphism $\psi : \mathrm{GL}_\infty(\mathbb{S}_1) \to \mathrm{GL}_\infty(L_1)$. By Proposition 2.10, the image of the group homomorphism $\det\circ\psi : \mathrm{GL}_\infty(\mathbb{S}_1) \stackrel{\psi}{\to} \mathrm{GL}_\infty(L_1) \stackrel{\det}{\to} L_1^*$ is K^*. Therefore, there is a well determined group epimorphism:

$$\overline{\det} := \det\circ\psi : \mathrm{GL}_\infty(\mathbb{S}_1) \to K^*. \tag{13}$$

By the very definition, $\overline{\det}(E_\infty(\mathbb{S}_1)) = 1$ and $\overline{\det}(\mu(\lambda)) = \lambda$ for all elements $\mu(\lambda) = \lambda E_{00}+1-E_{00} \in U(K)$ where $\lambda \in K^*$. Therefore, there is the exact sequence of groups:

$$1 \to E_\infty(\mathbb{S}_1) \to \mathrm{GL}_\infty(\mathbb{S}_1) \stackrel{\overline{\det}}{\to} K^* \simeq \mathrm{K}_1(\mathbb{S}_1) \to 1. \tag{14}$$

Corollary 2.12 *Each element a of the group $\mathrm{GL}_\infty(\mathbb{S}_1) = U(K) \ltimes E_\infty(\mathbb{S}_1)$ is a unique product $a = \mu(\lambda)e$ where $\mu(\lambda) \in U(K)$ and $e \in E_\infty(\mathbb{S}_1)$. Moreover, $\lambda = \overline{\det}(a)$ and $e = \mu(-\overline{\det}(a))a$.*

Recall that that $G_2 = S_2 \ltimes \mathbb{T}^2 \ltimes \mathrm{Inn}(\mathbb{S}_2)$ (Theorem 1.1.(1)) and $(1+\mathfrak{a}_2)^* \simeq \mathrm{Inn}(\mathbb{S}_2)$, $u \leftrightarrow \omega_u$ (Theorem 1.2.(3)). We *identify* the groups $(1+\mathfrak{a}_2)^*$ and $\mathrm{Inn}(\mathbb{S}_2)$ via $u \leftrightarrow \omega_u$.

Theorem 2.13 *1. $G_2 = S_2 \ltimes \mathbb{T}^2 \ltimes \Theta \ltimes ((U_1(K) \ltimes E_\infty(\mathbb{S}_1(2)) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(\mathbb{S}_1(1)))$.*

2. $G_2 \simeq S_2 \ltimes \mathbb{T}^2 \ltimes \mathbb{Z} \ltimes ((K^* \ltimes E_\infty(\mathbb{S}_1)) \boxtimes_{\mathrm{GL}_\infty(K)} (K^* \ltimes E_\infty(\mathbb{S}_1)))$.

Proof. By (10), Proposition 2.9, and Proposition 2.10,

$$(1+\mathfrak{a}_2)^* = \Theta \ltimes ((U_1(K) \ltimes E_\infty(\mathbb{S}_1(2)) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(\mathbb{S}_1(1))), \tag{15}$$

and the statements follow since $(1+F_2)^* \simeq \mathrm{GL}_\infty(K)$. $\square$

Corollary 2.14 $\mathbb{S}_2^* = K^* \times \Theta \ltimes ((U_1(K) \ltimes E_\infty(\mathbb{S}_1(2)) \boxtimes_{(1+F_2)^*} (U_2(K) \ltimes E_\infty(\mathbb{S}_1(1)))$.

Generators for the group G_2. Using Theorem 2.13 and (4), we can easily write down a set of generators for the group G_2:

$$\begin{aligned} s: &\quad x_i \mapsto x_{i+1}, y_i \mapsto y_{i+1}, \quad i=1,2;\\ t_{(\lambda,1)}: &\quad x_1 \mapsto \lambda x_1,\ \ y_1 \mapsto \lambda^{-1}y_1,\ \ x_2 \mapsto x_2,\ \ y_2 \mapsto y_2,\ \ \lambda \in K^*; \end{aligned}$$

ω_θ, $\omega_{\lambda E_{00}(1)+1-E_{00}(1)}$, $\omega_{\lambda E_{ij}(1)E_{kl}(2)+1-E_{ij}(1)E_{kl}(2)}$, $\omega_{1+\mu x_2^m E_{ij}(1)}$, $\omega_{1+\mu y_2^m E_{ij}(1)}$, $\omega_{1+\mu E_{ij}(1)}$ and where $\lambda \in K^*$, $\mu \in K$, $m, i, j, k, l \in \mathbb{N}$, $m \geq 1$, $i \neq j$ (note that $s\omega_{1+\lambda x_1^m E_{ij}(2)}s^{-1} = \omega_{1+\lambda x_2^m E_{ij}(1)}$, etc).

Each element $\sigma \in G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ is a unique product $st_\lambda\omega_u$ where $s \in S_n$, $t_\lambda \in \mathbb{T}^n$ and $\omega_u \in \mathrm{Inn}(\mathbb{S}_n)$. In [6], for each element $\sigma \in G_n$, using the elements $\sigma(x_i), \sigma(y_i) \in \mathbb{S}_n$, $i = 1, \ldots, n$ explicit algebraic formulae are found for the components s, t_λ, and ω_u of σ. So, the automorphism σ can be effectively (in finitely many steps) decomposed into the product $st_\lambda\omega_u$.

Proposition 2.15 $(1+F_2)^* \subseteq E_\infty(\mathbb{S}_1(i))$ *for* $i = 1, 2$.

Proof. Due to symmetry it suffices to show that $(1+F_2)^* \subseteq E_\infty(\mathbb{S}_1(2))$. Recall that the group $(1+F_2)^* = (1+\sum_{\alpha,\beta\in\mathbb{N}^2} KE_{\alpha\beta})$ is non-canonically isomorphic to the group $\mathrm{GL}_\infty(K)$. To see this we have to choose a bijection $b : \mathbb{N}^2 \to \mathbb{N}$. Then the matrix units $E_{\alpha\beta}$ can be seen as the usual matrix units $E_{b(\alpha)b(\beta)}$ and so $(1+F_2)^* \simeq \mathrm{GL}_\infty(K)$. This isomorphism depends on

the choice of the bijection. Since $(1+F_2)^* \simeq \mathrm{GL}_\infty(K)$, the group $(1+F_2)^*$ is generated by the elements $a = 1+\lambda E_{ij}(1)E_{kl}(2)$ where $\lambda \in K$ and $(i,k) \neq (j,l)$, and $b = 1+\lambda E_{00}(1)E_{00}(2)$ where $\lambda \in K\backslash\{-1\}$. It suffices to show that these generators belong to the group $E_\infty(\mathbb{S}_1(2))$.

First, let us show that $a \in E_\infty(\mathbb{S}_1(2))$. If $i \neq j$ then obviously the inclusion holds. If $i = j$, i.e. $a = 1+\lambda E_{ii}(1)E_{kl}(2)$, then necessarily $k \neq l$ since $(i,k) \neq (i,l)$. For an element g and h of a group, $[g,h] := ghg^{-1}h^{-1}$ is their *group commutator*. For any natural number l such that $l \neq i$, the elements $1+E_{il}(1)E_{kk}(2)$ and $1+\lambda E_{li}(1)E_{kl}(2)$ belong to the group $E_\infty(\mathbb{S}_1(2))$. Then so does their commutator

$$[1+E_{il}(1)E_{kk}(2), 1+\lambda E_{li}(1)E_{kl}(2)] = 1+\lambda E_{ii}(1)E_{kl}(2). \tag{16}$$

Therefore, all the generators a belong to the group $E_\infty(\mathbb{S}_1(2))$.

It remains to prove that $b \in E_\infty(\mathbb{S}_1(2))$. In the 2×2 matrix ring $M_2(\mathbb{S}_1(2))$ with entries in the algebra $\mathbb{S}_1(2)$ we have the equality, for all scalars $\lambda \in K\backslash\{-1\}$:

$$\begin{pmatrix} 1 & 0 \\ -\frac{y_2}{1+\lambda} & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\lambda x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\lambda^2 x_2}{1+\lambda} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}\begin{pmatrix} 1-\frac{\lambda E_{00}(2)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

This can be checked by direct multiplication using the equalities $y_2x_2 = 1$, $x_2y_2 = 1-E_{00}(2)$, $y_2E_{00}(2) = 0$ and $E_{00}(2)x_2 = 0$ in the algebra $\mathbb{S}_1(2)$. Since the first six matrices in the equality belong to the group $E_\infty(\mathbb{S}_1(2))$, the last matrix $c = \begin{pmatrix} 1-\frac{\lambda E_{00}(2)}{1+\lambda} & 0 \\ 0 & 1 \end{pmatrix}$ belongs to the group $E_\infty(\mathbb{S}_1(2))$ as well and can be written as

$$c = E_{00}(1)(1-\frac{\lambda}{1+\lambda}E_{00}(2)) + 1 - E_{00}(1) = 1-\frac{\lambda}{1+\lambda}E_{00}(1)E_{00}(2) \in (1+F_2)^*.$$

Since the map $\varphi : K\backslash\{-1\} \to K\backslash\{-1\}$, $\lambda \mapsto -\frac{\lambda}{1+\lambda}$, is a bijection ($\varphi^{-1} = \varphi$), all the elements b belong to the group $E_\infty(\mathbb{S}_1(2))$. The proof of the proposition is complete. $\square$

Proof of Theorem 1.2. The third statement follows at once from the second one. To prove the remaining two statements we use induction on n. The case when $n = 1$ is Theorem 4.6, [5]. So, let $n > 1$ and we assume that the first two statements hold for all natural numbers $n' < n$. Clearly, $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$ where $\mathbb{S}_1(i) := K\langle x_i, y_i\rangle \simeq \mathbb{S}_1$. Consider the following ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{p}_1 := F\otimes \mathbb{S}_{n-1},\ \mathfrak{p}_2 := \mathbb{S}_1 \otimes F \otimes \mathbb{S}_{n-2}, \ldots, \mathfrak{p}_n := \mathbb{S}_{n-1}\otimes F,\ \mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n.$$

Let $K(x_n)$ be the field of fractions of the polynomial algebra $K[x_n]$. It follows from the chain of algebra homomorphisms

$$\mathbb{S}_n \to \mathbb{S}_n/\mathfrak{p}_n \simeq \mathbb{S}_{n-1}\otimes \mathbb{S}_1/F \simeq \mathbb{S}_{n-1}\otimes K[x_n, x_n^{-1}] \to \mathbb{S}_{n-1}\otimes K(x_n) \simeq \mathbb{S}_{n-1}(K(x_n)) \tag{18}$$

and from the induction on n that $\mathbb{S}_n^* \subseteq \mathbb{S}_1(n) + \mathfrak{a}_n = \sum_{i\geq 1} Ky_n^i + K + \sum_{i\geq 1} Kx_n^i + \mathfrak{a}_n$ (since $\mathbb{S}_{n-1}(K(x_n))^* = K(x_n)^*$, by induction). By symmetry of the indices $1,\ldots,n$, we have the inclusion $\mathbb{S}_n^* \subseteq \bigcap_{j=1}^n (\sum_{i\geq 1} Ky_j^i + K + \sum_{i\geq 1} Kx_j^i + \mathfrak{a}_n) = K + \mathfrak{a}_n$ and so

$$K^*(1+\mathfrak{a}_n)^* \subseteq \mathbb{S}_n^* \subseteq \mathbb{S}_n^* \cap (K+\mathfrak{a}_n) = K^*\cdot(\mathbb{S}_n^*\cap(1+\mathfrak{a}_n)) = K^*(1+\mathfrak{a}_n)^* = K^*\times(1+\mathfrak{a}_n)^*$$

since $\mathfrak{a}_n$ is an ideal of the algebra $\mathbb{S}_n$ (and so $\mathbb{S}_n^*\cap(1+\mathfrak{a}_n) = (1+\mathfrak{a}_n)^*$). This proves statement 1.

It follows from statement 1, (18) and induction that $Z(\mathbb{S}_n) \subseteq K^*(1+\mathfrak{p}_n)^*$, hence, by symmetry,

$$Z(\mathbb{S}_n^*) \subseteq \bigcap_{i=1}^n K^*(1+\mathfrak{p}_i)^* = K^*(1+\bigcap_{i=1}^n \mathfrak{p}_i)^* = K^*(1+F_n)^*$$

where $F_n := \bigcap_{i=1}^n \mathfrak{p}_i$. Since $(1+F_n)^* \simeq \mathrm{GL}_\infty(K)$ (see Section 2, [4]) and the centre of the group $\mathrm{GL}_\infty(K)$ is $\{1\}$, statement 2 follows. $\square$

3 Normal subgroups of $\mathrm{GL}_\infty(\mathbb{S}_1)$ and the centres of the groups $\mathrm{GL}_\infty(\mathbb{S}_1)/\mathrm{SL}$ and $E_\infty(\mathbb{S}_1)/\mathrm{SL}$

In this section, several normal subgroups of the group $\mathrm{GL}_\infty(\mathbb{S}_1)$ are introduced, see (21) and Propositions 3.2.(4). The most important (and non-obvious) is the normal subgroup SL (Proposition 3.2.(4)). The group $(1+F_2)^*$ is *non-canonically* isomorphic to the group $\mathrm{GL}_\infty(K)$, hence it inherits the determinant homomorphism det, see (19), and $\mathrm{SL} := \{u \in (1+F_2)^* \mid \det(u) = 1\}$. We will show that the determinant det and the group SL *does not* depend on the isomorphism $(1+F_2)^* \simeq \mathrm{GL}_\infty(K)$. Moreover, the determinant is invariant under the conjugation of its argument by the elements of the group $\mathrm{GL}_\infty(\mathbb{S}_1)$, Proposition 3.2.(3). This is the central point of this section. It implies that the group SL is a *normal* subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$ and is a key fact in finding the centres of the groups $\mathrm{GL}_\infty(\mathbb{S}_1)/\mathrm{SL}$ and $E_\infty(\mathbb{S}_1)/\mathrm{SL}$ (Theorem 3.3).

In order to prove Theorem 3.3 we use results and notations of Section 2. In particular, we use the following group isomorphism:

$$(1+\mathfrak{p}_1)^* = (1+M_\infty(\mathbb{S}_1))^* \simeq \mathrm{GL}_\infty(\mathbb{S}_1), \;\; 1+\sum a_{ij}E_{ij}(1) \mapsto 1+\sum a_{ij}E_{ij},$$

where $a_{ij} \in \mathbb{S}_1 = \mathbb{S}_1(2)$ and E_{ij} are the matrix units. It is convenient to identify the groups $(1+\mathfrak{p}_1)^*$ and $\mathrm{GL}_\infty(\mathbb{S}_1)$ via this isomorphism, i.e. $E_{ij}(1) = E_{ij}$. Then, by Proposition 2.10,

$$\mathrm{GL}_\infty(\mathbb{S}_1) = U \ltimes E_\infty(\mathbb{S}_1)$$

where $U := \{\mu(\lambda) := \lambda E_{00} + 1 - E_{00} \mid \lambda \in K^*\} \simeq K^*$, $\mu(\lambda) \leftrightarrow \lambda$, and the groups $E_\infty(\mathbb{S}_1)$ and $(1+F_2)^*$ are normal subgroups of $\mathrm{GL}_\infty(\mathbb{S}_1)$.

As we have seen in the proof of Proposition 2.15 the group $(1+F_2)^*$ is isomorphic to the group $\mathrm{GL}_\infty(K)$. This isomorphism depends on the choice of the bijection b. For the group $\mathrm{GL}_\infty(K)$, we have the determinant (group epimorphism) $\det : \mathrm{GL}_\infty(K) \to K^*$, the short exact sequence of groups $1 \to \mathrm{SL}_\infty(K) \to \mathrm{GL}_\infty(K) \stackrel{\det}{\to} K^* \to 1$, and the decomposition $\mathrm{GL}_\infty(K) = U(K) \ltimes \mathrm{SL}_\infty(K)$ where $U(K) = \{\mu(\lambda) \mid \lambda \in K^*\}$. Therefore, for the group $(1+F_2)^*$ we have the determinant (group epimorphism) $\det : (1+F_2)^* \to K^*$, the short exact sequence of groups

$$1 \to \mathrm{SL} \to (1+F_2)^* \stackrel{\det}{\to} K^* \to 1 \qquad (19)$$

and the decomposition $(1+F_2)^* = U' \ltimes \mathrm{SL}$ where

$$\begin{aligned} U' &:= \{\mu'(\lambda) := \lambda E_{00}(1)E_{00}(2) + 1 - E_{00}(1)E_{00}(2) \mid \lambda \in K^*\} \simeq K^*, \;\; \mu'(\lambda) \leftrightarrow \lambda, \\ \mathrm{SL} &:= \{u \in (1+F_2)^* \mid \det(u) = 1\}. \end{aligned}$$

The group SL is generated by the elements $1+\lambda E_{\alpha\beta}$ where $\lambda \in K$ and $\alpha, \beta \in \mathbb{N}^2$ such that $\alpha \neq \beta$. Theorem 8.1, [5], says that the map $\det : (1+F_2)^* \to K^*$ *does not* depend on the choice of the bijection b.

Theorem 3.1 (Theorem 8.1, [5]) *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ be a finite dimensional vector space filtration on P_2 (i.e. $V_0 \subseteq V_1 \subseteq \cdots$ and $P_2 = \cup_{i\in\mathbb{N}} V_i$) and $a \in (1+F_2)^*$. Then $a(V_i) \subseteq V_i$ and $\det(a|_{V_i}) = \det(a|_{V_j})$ for all $i, j \gg 0$. Moreover, this common value of the determinants does not depend on the filtration $\mathcal{V}$ and, therefore, coincides with the determinant in (19).*

By Theorem 3.1, the group SL does not depend on the choice of the bijection b. The algebra $\mathbb{S}_2$ admits the *involution*:

$$\eta : \mathbb{S}_2 \to \mathbb{S}_2, \;\; x_i \mapsto y_i, \;\; y_i \mapsto x_i, \;\; i = 1, 2,$$

i.e. it is a K-algebra *anti-isomorphism* ($\eta(ab) = \eta(b)\eta(a)$ for all elements $a, b \in \mathbb{S}_2$) such that $\eta^2 = \mathrm{id}_{\mathbb{S}_2}$, the identity map on $\mathbb{S}_2$. It follows that

$$\eta(E_{ij}(k)) = E_{ji}(k)) \;\; \text{and} \;\; \eta(E_{\alpha\beta}) = E_{\beta\alpha} \qquad (20)$$

for all elements $i, j \in \mathbb{N}$, $k = 1, 2$, and $\alpha, \beta \in \mathbb{N}^2$. Therefore, $\eta(\mathfrak{p}_k) = \mathfrak{p}_k$ and $\eta(F_2) = F_2$. It is easy to see that $\eta((1+\mathfrak{p}_k)^*) = (1+\mathfrak{p}_k)^*$, $\eta((1+F_2)^*) = (1+F_2)^*$ and $\eta(E_\infty(\mathbb{S}_1(k))) = E_\infty(\mathbb{S}_1(k))$.

The polynomial algebra P_2 is equipped with the *cubic* filtration $\mathcal{C} := \{\mathcal{C}_m := \sum_{\alpha \in C_m} Kx^\alpha\}_{m\in\mathbb{N}}$ where $C_m := \{\alpha \in \mathbb{N}^2 \,|\, \text{all } \alpha_i \leq m\}$. The filtration $\mathcal{C}$ is an ascending, finite dimensional filtration such that $P_2 = \bigcup_{m\geq 0} \mathcal{C}_m$ and $\mathcal{C}_m\mathcal{C}_l \subseteq \mathcal{C}_{m+l}$ for all $m, l \geq 0$.

Proposition 3.2 *1. For all elements $a \in (1+F_2)^*$, $\det(\eta(a)) = \det(a)$.*

2. *Let $a \in \mathrm{GL}_\infty(\mathbb{S}_1) = (1+\mathfrak{p}_1)^*$ and let $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ be an ascending finite dimensional filtration on P_2 such that $a(V_i) \subseteq V_i$ for all $i \gg 0$. Then $\det(aba^{-1}) = \det(a)$ for all elements $b \in (1+F_2)^*$.*

3. *For all elements $a \in \mathrm{GL}_\infty(\mathbb{S}_1)$ and $b \in (1+F_2)^*$, $\det(aba^{-1}) = \det(a)$.*

4. *The group SL is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$. Moreover, each subgroup N of the group $(1+F_2)^*$ that contains the group SL is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$.*

Proof. 1. Recall that $(1+F_2)^* = U' \ltimes \mathrm{SL}$. It is obvious that $\eta(u) = u$ for all elements $u \in U'$ and $\eta(\mathrm{SL}) \subseteq \mathrm{SL}$ (by (20), the set of standard generators of the group SL is invariant under the action of the involution η). An element $a \in (1+F_2)^*$ is a unique product $a = us$ for some elements $u \in U'$ and $s \in \mathrm{SL}$. Now,

$$\det(\eta(a)) = \det(\eta(us)) = \det(\eta(s)u) = \det(u) = \det(a).$$

2. Applying Theorem 3.1 to the element $aba^{-1} \in (1+F_2)^*$ we have the result, for all $i \gg 0$:

$$\det(aba^{-1}) = \det(aba^{-1}|_{V_i}) = \det(a|_{V_i} \cdot b|_{V_i} \cdot a^{-1}|_{V_i}) = \det(b|_{V_i}) = \det(b).$$

3. We deduce statement 3 from the previous two. Recall that we have identified the groups $(1+\mathfrak{p}_1)^*$ and $\mathrm{GL}_\infty(\mathbb{S}_1)$. By Theorem 2.11, $\mathrm{GL}(\mathbb{S}_1) = U \ltimes E_\infty(\mathbb{S}_1)$. Since the subgroup $(1+F_2)^*$ of $\mathrm{GL}_\infty(\mathbb{S}_1)$ is normal, it suffices to show that statement 3 holds for generators of the groups U and $E_\infty(\mathbb{S}_1)$. Since $U = \{\mu(\lambda) \,|\, \lambda \in K^*\}$ and $\mu(\lambda)(\mathcal{C}_m) \subseteq \mathcal{C}_m$ for all $m \geq 0$ where $\mathcal{C} = \{\mathcal{C}_m\}_{m\in\mathbb{N}}$ is the cubic filtration on the polynomial algebra P_2, we see that $\det(\mu(\lambda)b\mu(\lambda)^{-1}) = \det(b)$, by statement 2. The group $E_\infty(\mathbb{S}_1)$ is generated by the elements (where $\lambda \in K$, $i \neq j$): $a_{ij} = 1 + E_{ij}\lambda x_2^n$ where $n \geq 1$; $b_{ij} = 1 + E_{ij}\lambda y_s^m$ where $m \geq 0$; and $1 + E_{ij}f$ where $f \in F_2$. Statement 3 holds for the elements $1 + E_{ij}f$ since they belong to the group $(1+F_2)^*$.

Since $b_{ij}(\mathcal{C}_k) \subseteq \mathcal{C}_k$ for all $k \gg 0$, statement 3 holds for the elements b_{ij}, by statement 2. Since $\eta(a_{ij}) = b_{ji}$, statement 3 holds for the elements a_{ij}, by statements 1 and 2. In more detail,

$$\det(a_{ij}ba_{ij}^{-1}) = \det(\eta(a_{ij}ba_{ij}^{-1})) = \det(b_{ji}^{-1}\eta(b)b_{ji}) = \det(\eta(b)) = \det(b).$$

4. By statement 3, the group SL is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$. If N is a subgroup of $(1+F_2)^*$ containing SL then $N = N' \ltimes \mathrm{SL}$ for a subgroup $N' = \{\mu(\lambda) \,|\, \lambda \in K'\}$ of U' where K' is an additive subgroup of the field K. Since $K' = \det(N)$ and $\det(aNa^{-1}) = \det(N) = K'$, by statement 3, we see that $aNa^{-1} \subseteq N$, i.e. N is a normal subgroup of $\mathrm{GL}_\infty(\mathbb{S}_1)$. $\square$

By Proposition 3.2.(4), there is a chain of normal subgroups of the group $\mathrm{GL}_\infty(\mathbb{S}_1)$:

$$\mathrm{SL} \subset (1+F_2)^* \subset E_\infty(\mathbb{S}_1) \subset \mathrm{GL}_\infty(\mathbb{S}_1). \tag{21}$$

Using the fact that $(1+F_2)^* = U' \ltimes \mathrm{SL}$, we have the chain of normal subgroups of the factor group $\mathrm{GL}_\infty(\mathbb{S}_1)/\mathrm{SL}$:

$$U' \subset E_\infty(\mathbb{S}_1)/\mathrm{SL} \subset \mathrm{GL}_\infty(\mathbb{S}_1)/\mathrm{SL}.$$

Theorem 3.3 *The group $U' \simeq K^*$ is the centre of both groups $\mathrm{GL}_\infty(\mathbb{S}_1)/\mathrm{SL}$ and $E_\infty(\mathbb{S}_1)/\mathrm{SL}$.*

Proof. By Proposition 3.2.(3), the group U' belongs to the centres of both groups. The algebra homomorphism $\mathbb{S}_1 \to \mathbb{S}_1/F \simeq K[x_2, x_2^{-1}]$ yields the group homomorphisms: $\mathrm{GL}_\infty(\mathbb{S}_1) \stackrel{\varphi}{\to} \mathrm{GL}_\infty(K[x_2, x_2^{-1}])$ and $E_\infty(\mathbb{S}_1) \stackrel{\psi}{\to} E_\infty(K[x_2, x_2^{-1}])$. By Proposition 2.10, $\mathrm{im}(\varphi) = U(K) \ltimes E_\infty(K[x_2, x_2^{-1}])$ and $\mathrm{im}(\psi) = E_\infty(K[x_2, x_2^{-1}])$. Since the groups $\mathrm{im}(\varphi)$ and $\mathrm{im}(\psi)$ have trivial centre and $\ker(\varphi) = \ker(\psi) = (1+F_2)^*/\mathrm{SL} = U' \ltimes \mathrm{SL}/\mathrm{SL} \simeq U'$, the result follows. $\square$

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Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk